# Energy Levels of $Tl^{208}$ and $Bi^{208}$†*

YEONG E. KIM
*Bell Telephone Laboratories, Murray Hill, New Jersey*

AND

JOHN O. RASMUSSEN
*University of California, Lawrence Radiation Laboratory Berkeley, California*

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The low-lying energy levels of $Tl^{208}$ and $Bi^{208}$ are calculated by using the $j$-$j$ coupling shell model and a residual Gaussian potential of Kim and Rasmussen, which was used in the shell-model calculation of $Bi^{210}$ and $Po^{210}$. The eigenvalues and eigenfunctions are presented and compared with experimental spectra. The calculated results agree rather well with available experimental data, and indicate that inclusion of the tensor force in the shell-model residual force is necessary in explaining the energy-level spectra of $Tl^{208}$ and $Bi^{208}$.

## I. INTRODUCTION

RECENTLY, it was demonstrated[1,2] that the tensor force is responsible for some low-energy nuclear properties of odd-odd spherical nuclei, $Y^{90}$, and $Bi^{210}$ and even-even $Po^{210}$. In particular, nuclear spectra of the ground-state multiplet in $Bi^{210}$ and RaE beta decay parameter $i\langle \mathbf{r}\rangle/\langle \boldsymbol{\sigma}\times\mathbf{r}\rangle$ have been explained by Kim and Rasmussen using a phenomenological Gaussian potential which explicitly includes the tensor force.[2]

It is interesting to see if one can obtain reasonable agreements with experimental data using the same potential for other neighboring odd-odd nuclei with a particle and a hole plus the doubly closed shell. In the present paper, we will consider two odd-odd nuclei, $Tl^{208}$ and $Bi^{208}$, and calculate the energy spectra of low-lying energy levels of these nuclei, using the $j$-$j$ coupling odd-group model with configuration mixing. The absolute energies of single-particle and single-hole states are obtained from empirical data, and no adjustment of the force parameters is attempted. Section II discusses the method of evaluating the matrix elements for the particle-hole interaction, and Sec. III will deal with zero-order energies. Finally, in Sec. IV the results of the calculation are presented and compared with the experimental data. Discussions are given in Sec. V.

## II. PARTICLE-HOLE INTERACTION

For the case of nuclei with the doubly closed-shell core plus one particle and one hole ($Tl^{208}$ and $Bi^{208}$), it is convenient to use the method of the second quantization. Brink and Satchler[3] showed that the occupation-number representation of Dirac[4] leads to a simpler procedure than the conventional one for the calculation of the matrix elements of operators in the shell model. The concept of particles and holes in the shell model in this representation was discussed thoroughly by Brink and Satchler, and some applications were made by Carter *et al.*[5] for calculations of the core-excited states in $Pb^{208}$. In the following, only a brief outline leading to the final expression for the matrix elements of the particle-hole interaction is given.

We define a vector for one-particle or one-hole plus the closed shell as

$$|(C+1)jm\rangle = \eta_{jm}^{\dagger}|C\rangle$$

or

$$|(C-1)j'm'\rangle = (-1)^{j'+m'}\eta_{j'-m'}|C\rangle,$$

respectively, where $|C\rangle$ represents the closed shell which is a spherically symmetric state with total angular momentum $J=0$. The operator $\eta_{jm}^{\dagger}$, the adjoint of $\eta_{jm}$, is the creation operator, which creates a particle in the single-particle state $|jm\rangle$ outside the closed shell when acting on $|C\rangle$. Similarly, $\eta_{j'm'}$ is the annihilation operator, which, when acting on $|C\rangle$, annihilates a particle in the state $|j'-m'\rangle$ inside the closed shell. For a system of fermions, these operators will have the following usual anticommutation relations

$$\{\eta_a^{\dagger},\eta_b^{\dagger}\}_+=0,$$
$$\{\eta_a,\eta_b\}_+=0,$$
$$\{\eta_{a'},\eta_a^{\dagger}\}_+=\delta_{a,a'}.$$

The phase factor $(-1)^{j'+m'}$ and the reversal of sign for $m$ are necessary for one-hole state because our basic single-particle states are spherically symmetric and hence we require that $\eta_{jm}^{\dagger}$ and $\eta_{jm}$ transform under finite rotation in the same way. The annihilation operator $\eta_{jm}$ transforms as the complex conjugate

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[1] Y. E. Kim, Phys. Rev. **131**, 1712 (1963).

[2] P. A. Mello and J. Flores, Nucl. Phys. **47**, 177 (1963); Y. E. Kim and John O. Rasmussen, Nucl. Phys. **47**, 184 (1963).

[3] D. M. Brink and G. R. Satchler, Nuovo Cimento **4**, 549 (1956).

[4] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, Oxford, England, 1958), Chap. X.

[5] J. C. Carter, W. T. Pinkston, and W. W. True, Phys. Rev. **120**, 504 (1960).

$(D_{m',m}{}^{j}(\alpha,\beta,\gamma))^*$ which is equal to $(-1)^{m'-m}D_{-m',-m}{}^{j}(\alpha,\beta,\gamma)$, so that if we choose $(-1)^{j+m}\eta_{j-m}$ instead of $\eta_{jm}$, it will transform under rotation as the $m$th row of $D_{m',m}{}^{j}(\alpha,\beta,\gamma)$ in the same way as $\eta_{jm}{}^{\dagger}$.

Instead of using the conventional method in which protons and neutrons are regarded as distinct particles, we may regard neutrons and protons as different states of the same fundamental particles by adopting the isobaric spin formalism, and require complete antisymmetry of the wave function with respect to exchange of isobaric spin as well as space and ordinary spin variables. This implies that the subscript of the operator $\eta^+$ and $\eta$ includes quantum number specifying the isobaric spin for the state in addition to the space and spin quantum numbers. This enables us to ensure the complete antisymmetry for the mixed system of two different nucleons, and we may still use the anticommutation relations given above.

The particle plus hole state will be constructed by taking a vector product of one-particle and one-hole states

$$|\bar{j}_1j_2JM\rangle=\sum_m(-1)^{j_1-m}\eta_{j_1m}\eta_{j_2M+m}{}^{\dagger}|\mathrm{C}\rangle \times(j_1-mj_2M+m|JM),$$

where the hole quantum numbers are distinguished by a bar. We are interested here in evaluating the matrix element of the two-body operator $V=\frac{1}{2}\sum_{i\neq j}v_{ij}$,

$$\begin{aligned}\langle\bar{j}_1j_2JM|V|\bar{j}_1'j_2'JM\rangle &=\sum_{m,m'}(-1)^{j_1-m+j_1'-m'}(j_1-mj_2M+m|JM)\\ &\quad\times(j_1'-m'j_2'M+m'|JM)\sum_{abcd}\langle ab|v|cd\rangle\\ &\quad\times\langle C|\eta_{j_2M+m}\eta_{j_1m}{}^{\dagger}\eta_a{}^{\dagger}\eta_b{}^{\dagger}\eta_d\eta_c\eta_{j_1'm'}\eta_{j_2'M+m'}{}^{\dagger}|C\rangle.\end{aligned}$$

After some manipulation of the creation and annihilation operators by using the anticommutation relation already mentioned, these sets of terms reduce to[5]

$$\begin{aligned}\langle\bar{j}_1j_2JM|V|\bar{j}_1'j_2'JM\rangle=&\sum_{j_c,j_c'}\tfrac{1}{2}\langle j_cj_c'JM|V|j_cj_c'JM\rangle\\ &+\sum_{j_c}[-\langle j_cj_1JM|V|j_cj_1JM\rangle\\ &+\langle j_cj_2JM|V|j_cj_2JM\rangle]\\ &+\langle\bar{j}_1j_2JM|i_{12}|\bar{j}_1'j_2'JM\rangle. \quad (1)\end{aligned}$$

The first term of these four sets of terms represents the total core energy, and we may consider this term as our zero point of energy. The second term represents the interaction of the core with one hole, and the third term is the interaction of the core with the one extra particle outside the closed shells. The minus sign of the second term can be physically understood if one remembers that the total core energy already included the interaction of the particle that is missing from the core with all the other particles in the core. The second and third terms are considered to be the single-hole or single-particle energies of the hole or particle, respectively, and are estimated from the single-hole or single-particle levels of neighboring nuclei and binding energies.

The last term $\langle\bar{j}_1j_2JM|i_{12}|\bar{j}_1'j_2'JM\rangle$ represents the particle-hole interaction and may be expressed as[3]

$$\begin{aligned}\langle\bar{j}_1j_2JM|i_{12}|\bar{j}_1'j_2'JM\rangle &=-\sum_k(-1)^{j_1+j_2+j_1'+j_2'}(2k+1)W(j_2'j_1j_1'j_2;kJ)\\ &\quad\times\langle j_1'j_2kq|v_{12}|j_1j_2'kq\rangle. \quad (2)\end{aligned}$$

Note the minus sign in front of the summation. It indicates that the particle-hole interaction may be regarded as repulsive for an attractive force. The method of evaluating the particle-particle matrix element appearing in (2) has been presented elsewhere.[1] In evaluating the particle-particle matrix element, the single-particle wave function $|jm\rangle$ is assumed to be

$$|jm\rangle=|(ls)jm\rangle=\sum(lm_lsm_s|jm)|lm_lsm_s\rangle.$$

### III. ZERO-ORDER ENERGIES

If one takes the $Pb^{208}$ core interaction energy as zero-point energy, then the ground-state energies are given by the separation energies

$$S(\mathrm{Tl}^{208}\text{ g.s.})=\mathrm{B.E.}(\mathrm{Tl}^{208})-\mathrm{B.E.}(\mathrm{Pb}^{208}),$$
$$S(\mathrm{Bi}^{208}\text{ g.s.})=\mathrm{B.E.}(\mathrm{Bi}^{208})-\mathrm{B.E.}(\mathrm{Pb}^{208}).$$

Similarly the single-hole or single-particle energies of the ground states are

$$-E_h(\mathrm{Tl}^{208}\text{ g.s.})=\mathrm{B.E.}(\mathrm{Pb}^{208})-\mathrm{B.E.}(\mathrm{Tl}^{207}),$$
$$E_p(\mathrm{Tl}^{208}\text{ g.s.})=\mathrm{B.E.}(\mathrm{Pb}^{209})-\mathrm{B.E.}(\mathrm{Pb}^{208}),$$
$$-E_h(\mathrm{Bi}^{208}\text{ g.s.})=\mathrm{B.E.}(\mathrm{Pb}^{208})-\mathrm{B.E.}(\mathrm{Pb}^{207}),$$
$$E_p(\mathrm{Bi}^{208}\text{ g.s.})=\mathrm{B.E.}(\mathrm{Bi}^{209})-\mathrm{B.E.}(\mathrm{Pb}^{208}),$$

where $E_h$ and $E_p$ are regarded to be just the second and third terms of (1),

$$E_h=\sum_{j_c}-\langle j_cj_1JM|V|j_cj_1JM\rangle_{\text{g.s.}}$$

and

$$E_p=\sum_{j_c}\langle j_cj_2JM|V|j_cj_2JM\rangle_{\text{g.s.}},$$

respectively. The interesting quantity is the particle-hole interaction energy $V_{\text{int}}$

$$V_{\text{int}}=S-(E_h+E_p),$$

which may be compared with the theoretical value of $\langle\bar{j}_1j_2JM|i_{12}|\bar{j}_1j_2JM\rangle_{\text{g.s.}}$ as will be shown later. Since $(E_h+E_p)$ is constant for a given nucleus, we will take the sum of the first, second, and third terms of (1) for the ground state as our zero-point energy. The single-hole and single-particle energies for the excited states will be expressed in this scale. For $Tl^{208}$ the neutron

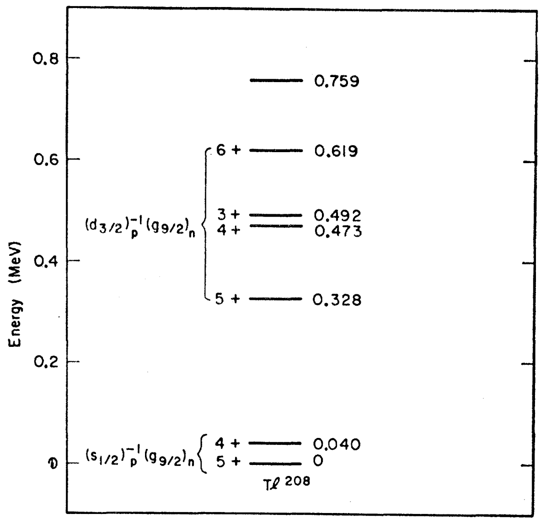


FIG. 1. Experimentally observed low-energy levels in $Tl^{208}$.

single-particle energies are taken from $Pb^{209}$ single-particle levels observed by Mukherjee and Cohen[6] and the proton single-hole levels are taken from $Tl^{207}$.[7] The resulting zeroth-order energies for $Tl^{208}$ are listed in Table I.

For $Bi^{208}$, the proton single-particle levels are taken from $Bi^{209}$,[8] and the neutron single-hole states are taken from $Pb^{207}$.[9] The resulting zeroth-order energies for $Bi^{208}$ are shown in Table II.

Six levels in $Tl^{208}$ were observed from the alpha decay of $Bi^{212}$. The alpha-gamma angular-correlation measurements of Horton and Sherr[10] and of Weale[11] suggest that the angular momenta of the ground state and the first excited states are 5 and 4, respectively,

TABLE I. Zeroth-order energies for $Tl^{208}$.

| Even-parity states Configuration (proton-neutron) (hole-particle) | Energy (MeV) | Odd-parity states Configuration (proton-neutron) (hole-particle) | Energy (MeV) |
|---|---|---|---|
| $s_{1/2}g_{9/2}$ | 0.0 | $s_{1/2}j_{15/2}$ | 1.41 |
| $d_{3/2}g_{9/2}$ | 0.37 | $d_{3/2}j_{15/2}$ | 1.78 |
| $s_{1/2}i_{11/2}$ | 0.77 | | |
| $d_{3/2}i_{11/2}$ | 1.14 | | |
| $s_{1/2}d_{5/2}$ | 1.56 | | |
| $d_{3/2}d_{5/2}$ | 1.93 | | |
| $s_{1/2}s_{1/2}$ | 2.03 | | |
| $d_{3/2}s_{1/2}$ | 2.40 | | |
| $s_{1/2}g_{7/2}$ | 2.47 | | |
| $s_{1/2}d_{3/2}$ | 2.52 | | |
| $d_{3/2}g_{7/2}$ | 2.84 | | |
| $d_{3/2}d_{3/2}$ | 2.89 | | |

[6] P. Mukherjee and B. L. Cohen, Phys. Rev. **127**, 1284 (1962).
[7] L. Silverberg, Arkiv Fysik **20**, 355 (1961).
[8] R. M. Hoff and J. M. Hollander, Phys. Rev. **109**, 447 (1958).
[9] D. E. Alburger and A. W. Sunyar, Phys. Rev. **99**, 695 (1955).
[10] J. Horton and R. Sherr, Phys. Rev. **90**, 388(A) (1953); J. Horton, *ibid.* **101**, 717 (1956).
[11] J. W. Weale, Proc. Phys. Soc. (London) **A68**, 35 (1955).

TABLE II. Zeroth-order energies for $Bi^{208}$.

| Even-parity states Configuration (neutron-proton) (hole-particle) | Energy (MeV) | Odd-parity states Configuration (neutron-proton) (hole-particle) | Energy (MeV) |
|---|---|---|---|
| $p_{1/2}h_{9/2}$ | 0.0 | $i_{13/2}h_{9/2}$ | 1.63 |
| $f_{5/2}h_{9/2}$ | 0.57 | $i_{13/2}f_{7/2}$ | 2.53 |
| $p_{3/2}h_{9/2}$ | 0.90 | | |
| $p_{1/2}f_{7/2}$ | 0.90 | | |
| $f_{5/2}f_{7/2}$ | 1.47 | | |
| $p_{3/2}f_{7/2}$ | 1.80 | | |
| $f_{7/2}h_{9/2}$ | 2.35 | | |
| $f_{7/2}f_{7/2}$ | 3.25 | | |

which is also consistent with the beta decay of the $Tl^{208}$ ground state to the excited states in $Pb^{208}$. The $Tl^{208}$ ground state decays predominantly into the 5− and 4− states of $Pb^{208}$ with $\log ft \sim 5.7$, but very weakly to the 3− state of $Pb^{208}$.[12,13] The 40-keV gamma transition in the ground-state doublet has been established by Graham and Bell[14] to be predominantly $M1$ from both the $L$-subshell conversion-electron intensity ratio ($L_I/L_{II}/L_{III}$) and lifetime. Spin and parity assignments for the observed levels in $Tl^{208}$ are presented in Fig. 1, and are consistent with the internal-conversion-coefficient measurements by Nielsen,[15] and more recent work by Emery and Kane.[16] The most recent work of alpha-gamma angular-correlation measurements by Cobb confirms these assignments shown in Fig. 1.[17]

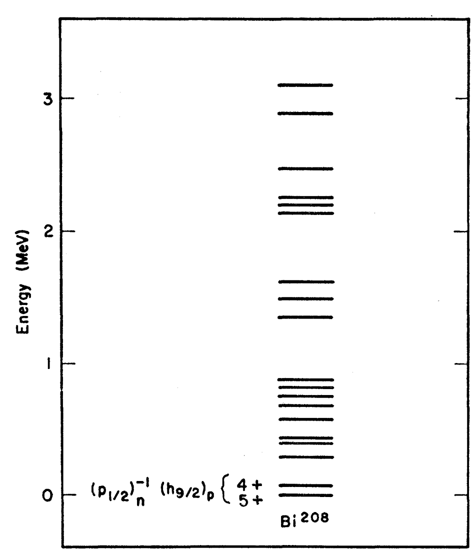


FIG. 2. Experimentally observed low-lying levels in $Bi^{208}$.

[12] L. G. Elliott, R. L. Graham, J. Walker, and J. L. Wolfson, Phys. Rev. **93**, 356 (1954).
[13] G. Schupp, H. Daniel, G. W. Eakins, and E. N. Jensen, Phys. Rev. **120**, 189 (1960).
[14] R. L. Graham and R. E. Bell, Can. J. Phys. **31**, 377 (1953).
[15] O. B. Nielsen, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 16 (1955).
[16] G. T. Emery and W. R. Kane, Phys. Rev. **118**, 755 (1960).
[17] W. C. Cobb, Phys. Rev. **132**, 1693 (1963).

TABLE III. Energy levels excited in $Bi^{209}(d,t)Bi^{208}$ reaction (Ref. 1).

| Excitation energy (MeV) | Relative yield at 45° | Excitation energy (MeV) | Relative yield at 45° |
|---|---|---|---|
| 0 | 104 | 1.35 | 6 |
| 0.07 | 84 | 1.49 | 5 |
| 0.29 | 29 | 1.62 | 5 |
| 0.40 | 72 | 2.14 | 15 |
| 0.43 | 53 | 2.20 | 25 |
| 0.58 | 19 | 2.25 | 6 |
| 0.68 | 60 | 2.47 | 8 |
| 0.75 | 65 | 2.89 | 2 |
| 0.82 | 18 | 3.10 | 4 |
| 0.88 | 122 | | |

From the shell-model calculation with a delta-function force, Pryce has interpreted the two lowest levels to be a doublet resulting from the splitting of the $(s_{1/2}g_{9/2})$ configuration.[18] Similarly, the four upper levels can be atrributed to the various spin states arising from the configuration $[(d_{3/2})^{-1}(g_{9/2})]$. Pryce's calculation disagrees slightly with the experimental level sequence shown in Fig. 1. The 3+ and 6+ states are inverted in his calculated results.

Recently, Mukherjee and Cohen have studied the low-energy spectrum of $Bi^{208}$ by the $(d,t)$ reaction on $Bi^{209}$.[6] Nineteen levels were resolved as shown in Fig. 2. Their experimental data on $Bi^{208}$ are summarized in Table III. Prior to this experiment, Duffield and Vegors found an isomeric state in $Bi^{208}$ with a lifetime of 2.7 msec from the $(\gamma,n)$ reaction on $Bi^{209}$.[19] This isomeric state cascades to the ground state by two gamma transitions of 921 and 509 keV. Partly from the internal-conversion-coefficient measurements and partly from Wahlborn's shell-model calculation with a delta-function force,[20] they proposed the following decay scheme:

$$10 \xrightarrow[921\text{ keV}]{E3} 7+ \xrightarrow[509\text{ keV}]{E2} 5+\,.$$

TABLE IV. Values of the force parameters used in $Bi^{210}$ and $Po^{210}$ calculations.

| Components | Strength (MeV) | Range (F) |
|---|---|---|
| Central triplet-even | −355.24 | 0.706 |
| Central singlet-even | −133.20 | 1.018 |
| Central triplet-odd | 0.0 | ··· |
| Central singlet-odd | 11.01 | 1.476 |
| Tensor triplet-even | −99.28 | 1.407 |
| Tensor triplet-odd | 9.50 | 1.845 |

[18] M. H. L. Pryce, Proc. Phys. Soc. (London) **A65**, 773 (1952).
[19] R. B. Duffield and S. H. Vegors, Jr., Phys. Rev. **112**, 1958 (1958).
[20] S. Wahlborn, Nucl. Phys. **3**, 644 (1957). Also see *Proceedings of the International Conference on Nuclear Structure at Weizmann Institute of Science, Rehovoth, Israel, 1957* (North-Holland Publishing Company, Amsterdam, 1958).

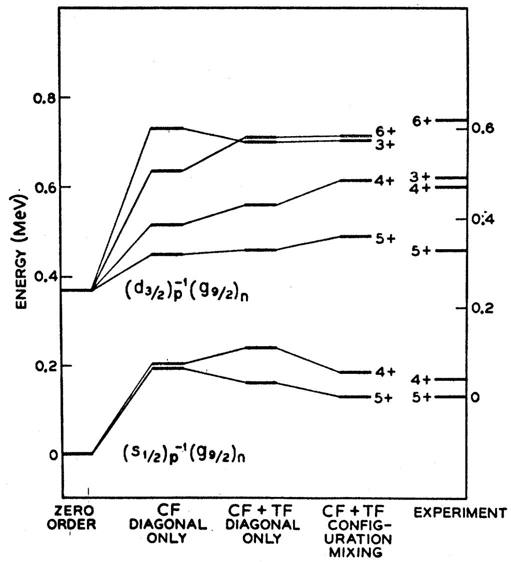


FIG. 3. Comparison of the experimental and calculated spectra of $Tl^{208}$. The abbreviations CF and TF refer to the central and tensor forces, respectively.

As will be shown later, our calculated results also offer a natural explanation of such a high-spin isomeric state. From the $(d,t)$ reaction work the ground state appears to be 5+ and the first excited state 4+, since the ground state has a higher relative cross section. Such assignment contradicts Wahlborn's calculation placing the 4+ at ground, but agrees with our 5+ ground state assignment.

Further experimental information on the first excited state of $Bi^{208}$ comes from work of Jones on alpha decay of the $At^{212}$ isomers.[21] The energy was determined as 63 keV.

Asaro and Perlman have studied the rare electron capture branching of $Po^{208}$ and have additional evidence on other excited states.[22] We discuss their work in a later section.

### IV. CALCULATED SPECTRA

The residual interaction $v_{12}$ appearing in (2) is chosen as

$$v_{12}=V^C(r_{12})+V^T(r_{12})S_{12}\,.$$

Here

$$\begin{aligned}V^C(r_{12})=[&V_{\mathrm{TE}}{}^CP_{\mathrm{TE}}\exp(-\beta_{\mathrm{TE}}{}^Cr_{12}{}^2)\\&+V_{\mathrm{SE}}{}^CP_{\mathrm{SE}}\exp(-\beta_{\mathrm{SE}}{}^Cr_{12}{}^2)\\&+V_{\mathrm{TO}}{}^CP_{\mathrm{TO}}\exp(-\beta_{\mathrm{TO}}{}^Cr_{12}{}^2)\\&+V_{\mathrm{SO}}{}^CP_{\mathrm{SO}}\exp(-\beta_{\mathrm{SO}}{}^Cr_{12}{}^2)]\,,\end{aligned}$$

and

$$\begin{aligned}V^T(r_{12})=[&V_{\mathrm{TE}}{}^TP_{\mathrm{TE}}\exp(-\beta_{\mathrm{TE}}{}^Tr_{12}{}^2)\\&+V_{\mathrm{TO}}{}^TP_{\mathrm{TO}}\exp(-\beta_{\mathrm{TO}}{}^Tr_{12}{}^2)]\,,\end{aligned}$$

[21] W. B. Jones, Phys. Rev. **130**, 2042 (1963).
[22] F. Asaro and I. Perlman, Lawrence Radiation Laboratory (unpublished results).

TABLE V. Calculated eigenvalues and energy levels in $Tl^{208}$. In the right column, eigenvalues are expressed in a new energy scale in which the ground state lies at zero energy. The indicated configuration is taken to be dominant.

| Configuration (proton-neutron) (hole-particle) | $J$ | Eigenvalues (MeV) | Energy (MeV) |
|---|---|---|---|
| $s_{1/2}g_{9/2}$ | 4+ | 0.183 | 0.053 |
| | 5+ | 0.130 | 0.0 |
| $d_{3/2}g_{9/2}$ | 3+ | 0.695 | 0.565 |
| | 4+ | 0.616 | 0.486 |
| | 5+ | 0.491 | 0.361 |
| | 6+ | 0.712 | 0.582 |
| $s_{1/2}i_{11/2}$ | 5+ | 0.832 | 0.702 |
| | 6+ | 0.951 | 0.821 |
| $d_{3/2}i_{11/2}$ | 4+ | 1.516 | 1.386 |
| | 5+ | 1.277 | 1.147 |
| | 6+ | 1.265 | 1.135 |
| | 7+ | 1.276 | 1.146 |
| $s_{1/2}d_{5/2}$ | 2+ | 1.732 | 1.602 |
| | 3+ | 1.790 | 1.660 |
| $d_{3/2}d_{5/2}$ | 1+ | 2.372 | 2.242 |
| | 2+ | 2.125 | 1.995 |
| | 3+ | 2.136 | 2.006 |
| | 4+ | 2.378 | 2.248 |
| $s_{1/2}s_{1/2}$ | 0+ | 2.878 | 2.748 |
| | 1+ | 3.005 | 2.875 |
| $d_{3/2}s_{1/2}$ | 1+ | 2.560 | 2.430 |
| | 2+ | 2.659 | 2.529 |
| $s_{1/2}g_{7/2}$ | 3+ | 2.554 | 2.424 |
| | 4+ | 2.676 | 2.546 |
| $s_{1/2}d_{3/2}$ | 1+ | 2.818 | 2.688 |
| | 2+ | 2.714 | 2.584 |
| $d_{3/2}g_{7/2}$ | 2+ | 3.250 | 3.120 |
| | 3+ | 3.015 | 2.885 |
| | 4+ | 2.961 | 2.831 |
| | 5+ | 2.996 | 2.866 |
| $d_{3/2}d_{3/2}$ | 0+ | 3.425 | 3.295 |
| | 1+ | 3.431 | 3.301 |
| | 2+ | 3.018 | 2.888 |
| | 3+ | 3.068 | 2.938 |
| $s_{1/2}j_{15/2}$ | 7− | 1.573 | 1.443 |
| | 8− | 1.498 | 1.368 |
| $d_{3/2}j_{15/2}$ | 6− | 2.029 | 1.899 |
| | 7− | 1.968 | 1.838 |
| | 8− | 1.885 | 1.755 |
| | 9− | 2.105 | 1.975 |

where $P_{\mathrm{TE}}$, $P_{\mathrm{SE}}$, $P_{\mathrm{TO}}$, and $P_{\mathrm{SO}}$ are the projection operators for the triplet-even, singlet-even, triplet-odd, and singlet-odd states, respectively, and the $V$'s are the corresponding strength parameters. The operator $S_{12}$ is the tensor-force operator defined as

$$S_{12}=[3(\boldsymbol{\sigma}_1\cdot\mathbf{r}_{12})(\boldsymbol{\sigma}_2\cdot\mathbf{r}_{12})]/r_{12}^2-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2.$$

The strength and range parameters $V$ and $\beta$, which were used by Kim and Rasmussen in the $Bi^{210}$ and $Po^{210}$ calculation, are presented in Table IV. The same parameters are used for $Tl^{208}$ and $Bi^{208}$ without any modifications.

The harmonic-oscillator radial wave function will be used throughout the numerical calculations with the harmonic-oscillator spacing $\hbar\omega=\hbar^2\nu/m\cong 41A^{-1/3}$ MeV.

The particle-hole matrix elements are calculated by the method described in Sec. II. The resulting matrix is then diagonalized to obtain the eigenvalues and eigenfunctions. In diagonalizing the matrix, the off-diagonal tensor-force matrix elements are neglected.

For $Tl^{208}$, the calculated results are schematically compared in Fig. 3. The eigenvalues are presented in Table V, and corresponding eigenfunctions are presented in Table VI for only the even-parity states of the lowest three configurations.

For $Bi^{208}$, the eigenvalues are presented in Table VII and the eigenfunctions for the even-parity states of the lowest four configurations are presented in Table VIII.

## V. DISCUSSION

For $Tl^{208}$, the agreement of calculated and experimental spectra as shown in Fig. 3 is good if one considers that the same potential used in $Bi^{210}$ and $Po^{210}$ was used without any modifications. The comparison of the other calculated levels with experiment is not feasible at present since no further experimental information is available. Although the tensor-force effects are not large in

TABLE VI. Calculated eigenfunctions for $Tl^{208}$.

| Eigenvalues (MeV) | Eigenfunctions $s_{1/2}g_{9/2}$ | $d_{3/2}g_{9/2}$ | $s_{1/2}i_{11/2}$ | $d_{3/2}i_{11/2}$ | $s_{1/2}d_{5/2}$ | $d_{3/2}d_{5/2}$ | $s_{1/2}g_{7/2}$ | $d_{3/2}g_{7/2}$ | $d_{3/2}d_{3/2}$ |
|---|---|---|---|---|---|---|---|---|---|
| $J=3$ | | | | | | | | | |
| 0.695 | | 0.9983 | | | −0.0449 | 0.0226 | 0.0184 | 0.0218 | −0.0018 |
| $J=4$ | | | | | | | | | |
| 0.183 | 0.9322 | 0.3613 | | −0.0058 | | 0.0152 | −0.0074 | −0.0146 | |
| 0.616 | −0.3613 | 0.9320 | | −0.0123 | | 0.0114 | −0.0136 | 0.0174 | |
| 1.516 | 0.0001 | 0.0124 | | 0.9971 | | 0.0007 | −0.0700 | −0.0259 | |
| $J=5$ | | | | | | | | | |
| 0.130 | 0.9538 | −0.3001 | −0.0022 | 0.0117 | | | | 0.0089 | |
| 0.491 | 0.3000 | 0.9532 | 0.0336 | 0.0087 | | | | −0.0082 | |
| 0.832 | −0.0121 | −0.0323 | 0.9472 | 0.3178 | | | | 0.0244 | |
| 1.277 | −0.0102 | 0.0053 | −0.3165 | 0.9474 | | | | −0.0453 | |
| $J=6$ | | | | | | | | | |
| 0.712 | | 0.9993 | −0.0307 | 0.0213 | | | | | |
| 0.951 | | 0.0365 | 0.9249 | −0.3785 | | | | | |
| 1.265 | | −0.0081 | 0.3790 | 0.9253 | | | | | |

TABLE VII. Calculated eigenvalues and energy levels in $Bi^{208}$. In the right column, eigenvalues are expressed in a new energy scale in which the ground state lies at zero energy. The indicated configuration is taken to be dominant.

| Configuration (neutron-proton) (hole-particle) | $J$ | Eigenvalues (MeV) | Energy (MeV) |
|---|---|---|---|
| $p_{1/2}h_{9/2}$ | 4+ | 0.142 | 0.081 |
| | 5+ | 0.061 | 0.0 |
| $f_{5/2}h_{9/2}$ | 2+ | 0.981 | 0.920 |
| | 3+ | 0.691 | 0.630 |
| | 4+ | 0.657 | 0.596 |
| | 5+ | 0.683 | 0.622 |
| | 6+ | 0.590 | 0.529 |
| | 7+ | 0.725 | 0.664 |
| $p_{3/2}h_{9/2}$ | 3+ | 1.107 | 1.046 |
| | 4+ | 1.042 | 0.981 |
| | 5+ | 0.977 | 0.916 |
| | 6+ | 1.140 | 1.079 |
| $p_{1/2}f_{7/2}$ | 3+ | 1.049 | 0.988 |
| | 4+ | 1.121 | 1.060 |
| $f_{5/2}f_{7/2}$ | 1+ | 2.185 | 2.124 |
| | 2+ | 1.762 | 1.701 |
| | 3+ | 1.730 | 1.669 |
| | 4+ | 1.766 | 1.705 |
| | 5+ | 1.616 | 1.555 |
| | 6+ | 2.080 | 2.019 |
| $p_{3/2}f_{7/2}$ | 2+ | 2.244 | 2.183 |
| | 3+ | 2.021 | 1.960 |
| | 4+ | 1.950 | 1.889 |
| | 5+ | 2.036 | 1.975 |
| $f_{7/2}h_{9/2}$ | 1+ | 2.911 | 2.850 |
| | 2+ | 2.592 | 2.531 |
| | 3+ | 2.543 | 2.482 |
| | 4+ | 2.530 | 2.469 |
| | 5+ | 2.478 | 2.417 |
| | 6+ | 2.545 | 2.484 |
| | 7+ | 2.434 | 2.373 |
| | 8+ | 2.698 | 2.637 |
| $f_{7/2}f_{7/2}$ | 0+ | 4.350 | 4.289 |
| | 1+ | 4.347 | 4.286 |
| | 2+ | 3.627 | 3.566 |
| | 3+ | 3.597 | 3.536 |
| | 4+ | 3.418 | 3.357 |
| | 5+ | 3.491 | 3.430 |
| | 6+ | 3.321 | 3.260 |
| | 7+ | 3.579 | 3.518 |
| $i_{13/2}h_{9/2}$ | 2− | 2.816 | 2.755 |
| | 3− | 1.864 | 1.803 |
| | 4− | 2.051 | 1.990 |
| | 5− | 1.883 | 1.822 |
| | 6− | 1.908 | 1.847 |
| | 7− | 1.902 | 1.841 |
| | 8− | 1.824 | 1.763 |
| | 9− | 1.968 | 1.907 |
| | 10− | 1.748 | 1.687 |
| | 11− | 2.300 | 2.239 |
| $i_{13/2}f_{7/2}$ | 3− | 3.116 | 3.055 |
| | 4− | 2.709 | 2.648 |
| | 5− | 2.688 | 2.627 |
| | 6− | 2.656 | 2.595 |
| | 7− | 2.606 | 2.545 |
| | 8− | 2.654 | 2.593 |
| | 9− | 2.566 | 2.505 |
| | 10− | 2.740 | 2.679 |

$Tl^{208}$, the gound-state doublet states (4+ and 5+) have their tensor-force matrix elements with favorable opposite signs, so that the tensor force tends to raise the energy of the 4+ state and lower that of the 5+ state, as shown in Fig. 3. The tensor force also plays a specific role in correcting the inversion of the 3+ and 6+ states of the $[(d_{3/2})^{-1}(g_{9/2})]$ configuration; a strengthened

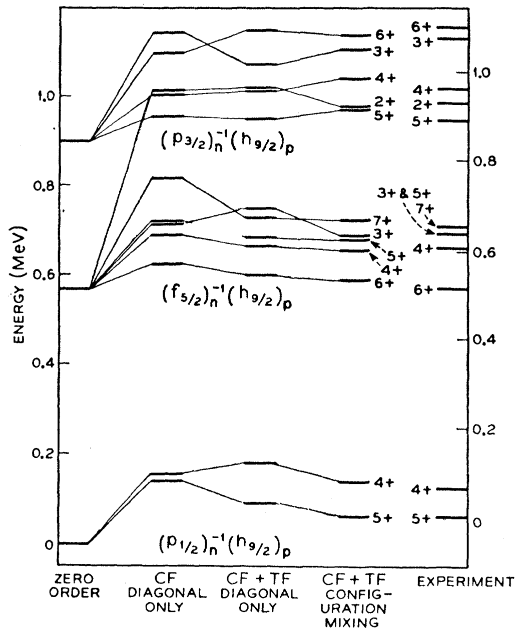


FIG. 4. Comparison of the experimental and calculated spectra of $Bi^{208}$. The symbols CF and TF stand for the central and tensor forces, respectively.

tensor force would improve the calculation with respect to several spacings.

For $Bi^{208}$, as in $Tl^{208}$, we obtain rather good agreement on the ground-state doublet. There has been some speculation as to whether the ground state is 4+ or 5+. The experimental relative cross sections obtained by Mukherjee and Cohen for the ground-state doublet as shown in Table III clearly suggest that the ground state is 5+, since the ratio of relative cross sections for the first excited state to the ground state is $84/104=0.807$, and this ratio is theoretically expected to be $[2(4)+1]/[2(5)+1]=0.818$ if the configurations are pure. The results of our calculation are very consistent with this experimental information. The calculation by Wahlborn[20] with a delta-function force gives the result that 4+ state is the ground state instead of the 5+ state. For our calculations the 5+ state comes lowest even if only the central force parts are included, as is seen from column 2 of Table IX. The energy difference is only 15 keV, and the tensor force makes the more significant contribution of 75 keV to increase the doublet splitting (see columns 2 and 3).

Most recently, higher excited states in $Bi^{208}$ are completely resolved in a high-resolution experiment $Bi^{209}(d,t)Bi^{208}$ by Erskine[23] which indicates that relative cross-section information provides clear assignments of spins and parities to these states. A comparison of our calculated spectra and the observed levels in $Bi^{208}$ by Erskine is schematically shown in Fig. 4. Our ex-

[23] J. R. Erskine, Argonne National Laboratory (private communication).

TABLE VIII. Calculated eigenfunctions for $Bi^{208}$.

| Eigenvalues (MeV) | Eigenfunctions $p_{1/2}h_{9/2}$ | $f_{5/2}h_{9/2}$ | $p_{3/2}h_{9/2}$ | $p_{1/2}f_{7/2}$ | $f_{5/2}f_{7/2}$ | $p_{3/2}f_{7/2}$ | $f_{7/2}h_{9/2}$ | $f_{7/2}f_{7/2}$ |
|---|---|---|---|---|---|---|---|---|
| $J=2$ | | | | | | | | |
| 0.981 | | 0.9872 | | | −0.0215 | −0.0156 | −0.1568 | −0.0122 |
| $J=3$ | | | | | | | | |
| 0.691 | | 0.9368 | 0.3471 | 0.0396 | 0.0001 | 0.0137 | 0.0135 | 0.0038 |
| 1.049 | | −0.0251 | −0.0372 | 0.9760 | −0.1213 | −0.1706 | 0.0195 | −0.0350 |
| 1.107 | | −0.3438 | 0.9303 | 0.0281 | 0.0373 | −0.0287 | −0.1148 | −0.0133 |
| $J=4$ | | | | | | | | |
| 0.142 | 0.9763 | −0.1806 | −0.1128 | −0.0025 | −0.0034 | −0.0019 | −0.0374 | −0.0059 |
| 0.657 | 0.1487 | 0.9594 | −0.2339 | 0.0220 | −0.0002 | −0.0109 | −0.0462 | −0.0046 |
| 1.043 | 0.1454 | 0.2080 | 0.9603 | −0.0582 | −0.1008 | −0.0152 | −0.0977 | −0.0102 |
| 1.121 | 0.0074 | −0.0087 | 0.0581 | 0.9727 | −0.1766 | 0.1358 | −0.0193 | 0.0166 |
| $J=5$ | | | | | | | | |
| 0.061 | 0.9806 | −0.1801 | 0.0739 | | −0.0071 | 0.0011 | 0.0203 | 0.0027 |
| 0.683 | 0.1532 | 0.9496 | 0.2713 | | −0.0088 | 0.0128 | 0.0320 | 0.0055 |
| 0.977 | −0.1175 | −0.2530 | 0.9587 | | 0.0237 | −0.0007 | −0.0506 | −0.0021 |
| $J=6$ | | | | | | | | |
| 0.590 | | 0.9920 | −0.1235 | | 0.0047 | | −0.0250 | −0.0016 |
| 1.140 | | 0.1200 | 0.9861 | | −0.0056 | | −0.1149 | −0.0087 |
| $J=7$ | | | | | | | | |
| 0.725 | | 0.9992 | | | | | 0.0404 | 0.0018 |

planation of the 2.7-msec isomeric state comes directly from the results of our calculation, and it involves the same spin sequence as originally suggested.[19] As shown in Fig. 5, the isomeric state is almost certainly the 10− state of the $[(i_{13/2})^{-1}(h_{9/2})]$ configuration, which may cascade through the 7+ state of the $[(f_{5/2})^{-1}(h_{9/2})]$ configuration to the ground state.

Asaro and Perlman have studied the gamma-ray spectrum associated with the rare electron capture of $Po^{208}$ to $Bi^{208}$, and they find a gamma ray of 285 keV in coincidence with a partially resolved doublet with energies ~570 and ~620 keV.[22] From the relative intensities of the gamma rays and the $\gamma-\gamma$ and $K-\gamma$ coincidence intensities, they infer an $M1$ character for the 285-keV transition. A brief examination of Fig. 4, showing the theoretical levels for $Bi^{208}$, suggests that the electron-capture branching of $Po^{208}$ goes by a second-forbidden transition to the lowest 2+ state, thence by an $M1$ transition to the 3+ state of the same multiplet. The 3+ state could decay to both the 5+ ground state and the 4+ first excited state. They compare the experimental and theoretical level energies as shown in Table X.

For both $Tl^{208}$ and $Bi^{208}$, the calculated results in Tables V and VII for the configurations involving $j_1$ or $j_2=\frac{1}{2}$ are consistent with de-Shalit and Walecka's coupling rule[24] except $[(s_{1/2})_p^{-1}(d_{5/2})_n]^{J=2,3}$, $[(s_{1/2})_p(s_{1/2})_n]^{J=0,1}$, and $[(s_{1/2})_p^{-1}(d_{3/2})_n]^{J=1,2}$ configurations in $Tl^{208}$. For the configurations involving $j_1$ and $j_2\geq\frac{3}{2}$, the calculated results are consistent with a weak-coupling rule that there is a tendency for the spin of the lowest state of a

TABLE IX. Energies for the ground-state doublet in $Bi^{208}$.

| Level spin and parity | Diagonal matrix elements (MeV) CF | TF | CF+TF | Final eigenvalues with config. mixing (MeV) |
|---|---|---|---|---|
| 4+ | 0.153 | 0.028 | 0.181 | 0.142 |
| 5+ | 0.138 | −0.047 | 0.091 | 0.061 |

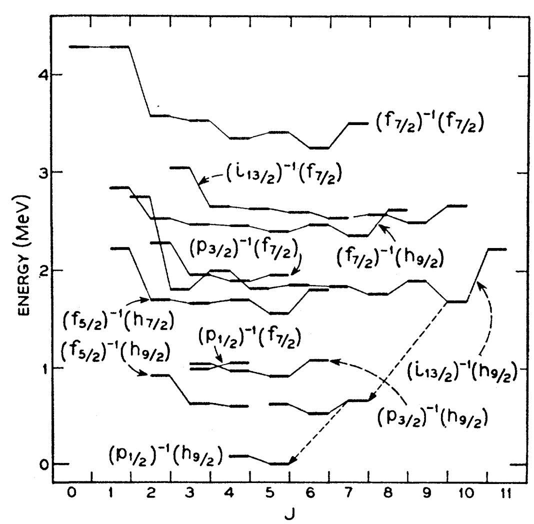


FIG. 5. Calculated energy levels of $Bi^{208}$. For each spin the left column lists the odd-parity states, and the right column the even-parity states. The various spin-$J$ states arising from the same configuration are connected by lines, and possible $E3$ and $E2$ transitions from the isomeric state $[(i_{13/2})^{-1}(h_{9/2})]^{J=10-}$ are shown by arrows and dashed lines.

[24] A. de-Shalit and J. D. Walecka, Nucl. Phys. 22, 184 (1961).

TABLE X. Energies of some $Bi^{208}$ excited states.

| Spin and parity | $E_{\text{theo}}$ (MeV) (Wahlborn)[a] | $E_{\text{theo}}$ (MeV) (This paper) | $E_{\text{exp}}$ (MeV) | Ref. |
|---|---|---|---|---|
| 5+ | 0.06 | 0.0 | 0.0 | |
| 4+ | 0.0 | 0.080 | 0.063 | 21 |
| 3+ | 0.62 | 0.630 | ~0.620 | 22 |
| 2+ | | 0.920 | ~0.905 | 22 |
| 7+ | | 0.664 | 0.509 | 19 |
| 10− | | 1.687 | 1.403 | 19 |

[a] Ref. 20.

given configuration to be given by[25]

$$J=j_1+j_2-1\,,$$

with a few exceptions of $[(d_{3/2})_p^{-1}(i_{11/2})_n]^{J=6}$ and $[(d_{3/2})_p^{-1}(d_{5/2})_n]^{J=3}$ in $Tl^{208}$.

With the $Tl^{208}$ eigenfunctions of Table VI we may re-examine the question of the lifetime of the 40-keV excited state. The experimental measurement of Siekman and de Waard[26] gives a half-life of $(2.6\pm1.0)\times10^{-12}$ sec, from which they deduce a mean life for photon emission $\tau_\gamma$ of $(1.2\pm0.5)\times10^{-10}$ sec de-Shalit[27] calculated theoretically a mean life $\tau_\gamma$ of $1.8\times10^{-10}$ sec for pure $(s_{1/2}g_{9/2})_I$ configurations. Using our mixed wave functions of Table VI we find that the substantial configuration admixture of $(d_{3/2}g_{9/2})_4$ into the 40-keV state has the effect of slowing down the transition by about 26% below the pure $(s_{1/2}g_{9/2})$ estimate. The configuration admixture of $(s_{1/2}g_{7/2})_4$ is about five times smaller than de-Shalit's estimate and would result in a negligible enhancement in the transition rate. The net result of configuration mixing is a slight slow-down from the pure $(s_{1/2}g_{9/2})$ estimate, a correction in the wrong direction to help match theory with experiment, but the discrepancy is still not very large.

A note of caution is in order regarding use of the wave functions of Tables VI and VIII. Tensor-force contributions to diagonal matrix elements were included, but because of computation time limitations the off-diagonal tensor contributions were not computed. This approximation is probably unimportant, so far as eigenvalues are concerned. Where eigenfunctions are concerned, the approximation may be very poor for some states; the case of the $Bi^{210}$ ground state, where the tensor-force off-diagonal contribution to the most important matrix element was of larger magnitude and opposite sign to the central force contribution, is a dramatic warning in this regard.

Another interesting comparison is the particle-hole interaction energies. The experimental particle-hole interaction energies, $V_{\text{int}}$, for $Tl^{208}$ and $Bi^{208}$ can be obtained by using various values of binding energies. Using binding energies from the table compiled by Wapstra *et al.*,[28] we obtain

$$V_{\text{int}}(Tl^{208}\text{ g.s.})=0.100\text{ MeV}\,,$$

and

$$V_{\text{int}}(Bi^{208}\text{ g.s.})=0.050\text{ MeV}\,,$$

which can be compared with the theoretical values of 0.130 and 0.061 MeV, respectively.

Although we do not believe that our choice of the residual force is necessarily the best one, the reasonable agreement with data of our calculation for $Tl^{208}$ and $Bi^{208}$ using identically the same $n-p$ force as deduced by fitting the $Bi^{210}$ spectrum lend encouragement to a view that the shell-model residual force may not be very different from the free two-nucleon force and that we may hope to find a residual force that can be used without modification for different nuclei.

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